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CONTRIBUTION TO THE THEORY OF ANALYTIC ALMOST PERIODIC FUNCTIONS

ON THE BEHAVIOUR OF AN ANALYTIC ALMOST PERIODIC FUNCTION IN THE NEIGHBOURHOOD OF A BOUNDARY FOR ITS ALMOST PERIODICITY

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CHAPTER I.

Introductory remarks.

We begin by recalling some of the fundamental notions and theorems in the theory of the almost periodic functions of a real as well as of a complex variable. We wish, however, essentially to confine ourselves to what is indispensable in the following[1].

A (complex) function $F(t)$ of a real variable t, continuous in $-\infty < t < \infty$, is called almost periodic, if to any given $\varepsilon > 0$ there exists a relatively dense set of translation numbers $\tau = \tau(\varepsilon)$, i. e. of numbers τ which satisfy the inequality

$$|F(t+\tau) - F(t)| \leqq \varepsilon \quad \text{in} \quad -\infty < t < \infty.$$

Here, a set of real numbers is called relatively dense, if there exists a length L such that any interval of length L contains at least one number of the set.

An almost periodic function $F(t)$ is bounded and uniformly continuous in $-\infty < t < \infty$, and the sum (and product) of two almost periodic functions proves again to be almost periodic.

[1] In its main features, the theory of the almost periodic functions was developed by the author in three papers in Acta Mathematica (vol. 45, 46, 47) under the common title "Zur Theorie der fastperiodischen Funktionen". Especially the last of these articles dealing with the functions of a complex variable is of importance for the present paper. Furthermore, we shall also make use of some of Bochner's results given in his important paper (Beiträge zur Theorie der fastperiodischen Funktionen, I. Teil, Math. Ann. vol. 96). However, it is not supposed that the papers are known to the reader and, therefore, the theorems applied will be directly formulated. For further information, cf. one of the monographs, A. S. Besicovitch: Almost periodic functions, Cambridge 1932; J. Favard: Leçons sur les fonctions presque-périodiques, Paris 1933; H. Bohr: Fastperiodische Funktionen, Berlin 1932.

For an arbitrary fixed real τ, let us consider the quantity

$$v(\tau) = \underset{-\infty < t < \infty}{\text{u. b.}} \left| F(t+\tau) - F(t) \right|.$$

This quantity, by the author originally denoted as the minimum error (corresponding to the given τ) of the function, was studied in detail by BOCHNER as a function of τ and, now, the function $v(\tau), -\infty < \tau < \infty$, is usually called the translation function of the given function $F(t)$. The function $v(\tau)$ is again an almost periodic function and its own translation function. A set $\{F(t)\}$ of almost periodic functions can be "majorised", if there exists an almost periodic function $F_0(t)$, which is denoted a majorant of the set $\{F(t)\}$, with the property that the translation function $v(\tau) = v_F(\tau)$ of any given function $F(t)$ of the set satisfies for all τ the inequality

$$v(\tau) \leqq v_0(\tau),$$

where $v_0(\tau)$ denotes the translation function of $F_0(t)$. A necessary and sufficient condition that the set of almost periodic functions $\{F(t)\}$ can be majorised is that the functions of the set are "uniformly" uniformly continuous and uniformly almost periodic. Here, a set of almost periodic functions is said to be uniformly almost periodic, if to any $\varepsilon > 0$ there exists a relatively dense set of numbers τ which are translation numbers corresponding to ε for any function of the set. A finite set of almost periodic functions can always be majorised. From this it becomes obvious that the sum of a finite number of almost periodic functions is again almost periodic.

A principal theorem in the theory of almost periodic functions of a real variable states that the class of all almost periodic functions is identical with the class of functions which, uniformly for all t, can be approximated by finite sums of the form $\sum_{1}^{N} a_n e^{i\lambda_n t}$, where the coefficients a_n are complex numbers, while the exponents λ_n are real numbers. With any almost periodic function $F(t)$ there is associated a Fourier series

$$F(t) \sim \sum A_n e^{i\Lambda_n t},$$

where the exponents Λ_n form the countably infinite set of values of λ for which the mean value

$$a(\lambda) = M\{F(t)e^{-i\lambda t}\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F(t)\, e^{-i\lambda t} dt$$

differs from zero, while the coefficients A_n are the corresponding values of $a(\Lambda_n)$. Starting from the Fourier series of an almost periodic function, by suitable summation methods finite sums of the form $\sum_1^N B_n e^{i\Lambda_n t}$ can be deduced (all exponents of which are Fourier exponents of $F(t)$) which converge uniformly to $F(t)$.

If $F(t)$ is almost periodic with the Fourier series $\sum A_n e^{i\Lambda_n t}$, and p is an arbitrary positive number, the (possibly empty) series $\sum' A_n e^{i\Lambda_n t}$ consisting of just the terms $A_n e^{i\Lambda_n t}$ in the original series with integral multipla of $\frac{2\pi}{p}$ as exponents, is the Fourier series of a continuous, purely periodic function $P(t)$. This function $P(t)$, which was especially considered by Bochner, will be denoted as the "periodic component of $F(t)$ belonging to the period p". This component can also be determined from the sequence of almost periodic functions

$$F_n(t) = \frac{F(t+p) + F(t+2p) + \cdots + F(t+np)}{n},$$

as this sequence $F_n(t)$ converges for $n \to \infty$ to $P(t)$, uniformly in the whole interval $-\infty < t < \infty$. From the limit equation $P(t) = \lim F_n(t)$ it results immediately that

$$\underset{-\infty<t<\infty}{\text{u. b.}} |P(t)| \leq \underset{-\infty<t<\infty}{\text{u. b.}} |F(t)|.$$

Let us now briefly recall some notions and theorems concerning the almost periodic functions of a complex variable.

A function $f(s) = f(\sigma + it)$, analytic in a vertical strip $\alpha < \sigma < \beta$ $(-\infty \leq \alpha < \beta \leq \infty)$, is called almost periodic in (α, β), if to any $\varepsilon > 0$ there exists a relatively dense set of translation numbers $\tau = \tau(\varepsilon)$ satisfying the inequality

$$|f(s+i\tau) - f(s)| \leq \varepsilon \quad \text{in the strip} \quad \alpha < \sigma < \beta.$$

In other words, we require that for any fixed σ in the interval $\alpha < \sigma < \beta$ the function $F_\sigma(t) = f(\sigma + it)$ is an almost periodic function of the real variable t, and that the functions of the set $\{F_\sigma(t)\}$ corresponding to $\alpha < \sigma < \beta$ are uniformly almost periodic. But, generally, this set cannot be majorised, since the functions (on account of the behaviour near the boundary of the strip) need not be "uniformly" uniformly continuous functions of t.

A function $f(s)$, analytic in $\alpha < \sigma < \beta$ $(-\infty \leqq \alpha < \beta \leqq \infty)$, is called almost periodic in $[\alpha, \beta]$, if it is almost periodic in every substrip (α_1, β_1) where $\alpha < \alpha_1 < \beta_1 < \beta$. We also use mixed brackets and thus speak of a function almost periodic in $[\alpha, \beta)$.

With each function $f(s)$, almost periodic in $[\alpha, \beta]$, is associated a Dirichlet series

$$f(s) \sim \sum A_n e^{\Lambda_n s}$$

with real exponents Λ_n which, for any fixed σ in $\alpha < \sigma < \beta$, gives the Fourier series of the almost periodic function $F_\sigma(t) = f(\sigma + it)$ of the real variable t. The set of all functions, almost periodic in $[\alpha, \beta]$, is identical with the set of the functions which can be approximated uniformly in $[\alpha, \beta]$ (i. e. uniformly in every substrip $\alpha_1 < \sigma < \beta_1$) by finite sums of the form $\sum a_n e^{\lambda_n s}$, where the λ_n are real numbers. If $f_1(s)$ and $f_2(s)$ are two analytic functions which both are almost periodic in $[\alpha, \beta]$, their sum $f_1(s) + f_2(s)$ is again almost periodic in $[\alpha, \beta]$. A corresponding general theorem does not hold for functions almost periodic in (α, β). Thus, the functions e^s and $e^{s\sqrt{2}}$ are both almost periodic (even purely periodic) in $(-\infty, \infty)$, whereas their sum $f(s)$ is almost periodic in $(-\infty, \infty]$, but not in $(-\infty, \infty)$; for any real $\tau \neq 0$ it is even valid that $|f(s+i\tau) - f(s)| \to \infty$ for $\sigma \to \infty$, uniformly in t. Another simple example is given by the two geometrical series

$$f_1(s) = \sum_{n=0}^{\infty} e^{ns} = \frac{1}{1-e^s} \quad \text{and} \quad f_2(s) = \sum_{n=0}^{\infty} e^{\sqrt{2}\,ns} = \frac{1}{1-e^{\sqrt{2}s}},$$

which both are almost periodic (even purely periodic) in $(-\infty, 0)$, while their sum is almost periodic in $(-\infty, 0]$, but not in $(-\infty, 0)$; for, as the function $f(s) = f_1(s) + f_2(s)$ has poles in (and only in) all points of the two arithmetic progressions $2m\pi i$ and $\sqrt{2}\,m\pi i$, there does not exist any number $\tau \neq 0$ for which the difference $f(s+i\tau)-f(s)$ is bounded in the whole half plane $-\infty<\sigma<0$ (as the set of poles is not transformed into itself by any translation) so that $f(s)$ for no ε has other translation numbers in $(-\infty, 0)$ than the trivial $\tau=0$.

It is the aim of this paper to investigate the behaviour of a function $f(s)$, almost periodic in a strip (α, β) (and not only in $[\alpha, \beta])$, in the immediate neighbourhood of one of the boundaries of the strip, for instance the right one. We shall assume the strip cut off (if necessary) on the left in order to avoid any difficulties on the left boundary of the strip.

For our purpose it will be convenient also to introduce the notion of a function almost periodic in $\{\alpha, \beta\}$ (and correspondingly in $\{\alpha, \beta)$ etc.). For $-\infty<\alpha<\beta<\infty$, the function $f(s)$ is called almost periodic in $\{\alpha, \beta\}$, if $f(s)$ is continuous in the closed strip $\alpha \leqq \sigma \leqq \beta$ and analytic and almost periodic in the open strip (α, β). Besides, it is clear that every translation number $\tau=\tau(\varepsilon)$ corresponding to $f(s)$ in the open strip (α, β) as a matter of course is a translation number $\tau(\varepsilon)$ for $f(s)$ in the closed strip $\{\alpha, \beta\}$, as the inequality

$$|f(s+i\tau)-f(s)| \leqq \varepsilon \quad \text{in} \quad \alpha<\sigma<\beta,$$

on account of the continuity of $f(s)$ in $\alpha \leqq \sigma \leqq \beta$, involves the inequality

$$|f(s+i\tau)-f(s)| \leqq \varepsilon \quad \text{in} \quad \alpha \leqq \sigma \leqq \beta.$$

By simple conclusions (known from the theory of the almost periodic functions of a real variable) we realize immediately that a function almost periodic in $\{\alpha, \beta\}$ is bounded and uniformly continuous in $\alpha \leqq \sigma \leqq \beta$. Thus, if $f(s)$ is almost periodic in $\{\alpha, \beta\}$, the functions $F_\sigma(t)=f(\sigma+it)$ $(\alpha \leqq \sigma \leqq \beta)$ are not only uniformly almost periodic, but also "uniformly" uniformly continuous, i. e. the set $\{F_\sigma(t)\}$ $(\alpha \leqq \sigma \leqq \beta)$ can be majorised. From this, it readily results that the sum of two functions $f_1(s)$ and $f_2(s)$, both almost periodic in $\{\alpha, \beta\}$, again is almost periodic in $\{\alpha, \beta\}$. We add, without going into details, that it is easy to prove that, if $f_1(s)$ is almost periodic in $\{\alpha, \beta\}$ and $f_2(s)$ is almost periodic in $\{\alpha, \beta)$, the sum $f_1(s)+f_2(s)$ is almost periodic in $\{\alpha, \beta)$ and not only in $\{\alpha, \beta]$.

But it is important to call attention to the fact that the sum of two functions, almost periodic in $\{\alpha, \beta)$, generally is almost periodic only in $\{\alpha, \beta]$, but not in $\{\alpha, \beta)$; thus, the two functions mentioned above, $\frac{1}{1-e^s}$ and $\frac{1}{1-e^{\sqrt{2}s}}$, are both almost periodic, for instance in $\{-1, 0)$, whereas the sum is not.

In the present paper, we shall study the functions, almost periodic in a strip of the type $\{\alpha, \beta)$, especially the unbounded functions of this type.

Of special importance for our investigation of a function almost periodic in $\{\alpha, \beta)$ is the set F of all translation numbers τ (where no ε is prescribed), i. e. the set of all real numbers τ for which the difference $f(s+i\tau)-f(s)$ is bounded in $\alpha \leq \sigma < \beta$. While, for a function almost periodic in $\{\alpha, \beta\}$, this set F consists of all real numbers, this is not necessarily the case for a function almost periodic in $\{\alpha, \beta)$; thus for the function $\frac{1}{1-e^s}$, purely periodic in $\{-1, 0)$, the set F consists of the numbers $2m\pi$. It will be shown below that the set F consists of all real numbers only in the trivial case, where the function $f(s)$ itself is bounded in $\{\alpha, \beta)$. Let us call the function

$$v(\tau) = \underset{s \text{ in } \{\alpha, \beta)}{\text{u. b.}} |f(s+i\tau)-f(s)|,$$

which is defined in the set F, the translation function of $f(s)$ in $\{\alpha, \beta)$. The set F contains together with τ also $-\tau$, and we have $v(-\tau) = v(\tau)$; moreover F contains together with τ_1 and τ_2 also $\tau_1+\tau_2$, and the inequality $v(\tau_1+\tau_2) \leq v(\tau_1)+v(\tau_2)$ is valid. The set F is a module, as it contains together with τ_1 and τ_2 also $\tau_1-\tau_2$ and it contains other numbers than 0 (as a subset it contains for instance the relatively dense set of all numbers $\tau = \tau(1)$); we shall denote F the "translation module" of the function $f(s)$ in $\{\alpha, \beta)$.

Moreover, the process mentioned above of separating the periodic component with a given period from an almost periodic function can be transferred from functions of a real variable to functions of a complex variable. For a function $f(s)$, almost periodic in $\{\alpha, \beta)$, and an arbitrarily chosen $p > 0$, we find that the sequence of the functions (evidently almost periodic in $\{\alpha, \beta]$)

$$f_n(s) = \frac{f(s+ip)+f(s+2\,ip)+\cdots+f(s+nip)}{n}$$

converges for $n \to \infty$ in the whole strip $\{\alpha, \beta)$ and even uniformly in each substrip $\{\alpha, \gamma\}$, where $\alpha < \gamma < \beta$, to a function $p(s)$ purely periodic with the period ip in $\{\alpha, \beta)$ whose Dirichlet series (Laurent series) $\sum B_m e^{m\frac{2\pi}{p}s}$ consists just of those terms $A_n e^{\Lambda_n s}$ in the Dirichlet series of the given function $f(s)$ the exponents Λ_n of which are integral multipla of $\frac{2\pi}{p}$. The transition from the real to the complex case is immediate, if we only observe that uniform convergence of the sequence $f_n(s)$ on the two lines $\sigma = \alpha$ and $\sigma = \gamma$ involves that $f_n(s)$ converges uniformly in the whole strip $\alpha \leq \sigma \leq \gamma$, as, according to the theorem of PHRAGMEN-LINDELÖF, u. b. $|f_{n_1}(s) - f_{n_2}(s)|$ remains unchanged, whether s varies in the whole strip $\alpha \leq \sigma \leq \gamma$ or only on the two boundaries $\sigma = \alpha$ and $\sigma = \gamma$ of the strip. Furthermore, if $f(s)$ is bounded in $\{\alpha, \beta)$, the component $p(s)$ is also bounded, and the inequality

$$\underset{s \text{ in } \{\alpha,\beta)}{\text{u. b.}} |p(s)| \leq \underset{s \text{ in } \{\alpha,\beta)}{\text{u. b.}} |f(s)|$$

is valid.

After this introducing Chapter I, the present paper is divided in three Chapters.

In Chapter II, the case $\beta = \infty$, i. e. the functions almost periodic in $\{\alpha, \infty)$, is treated. If the function $f(s)$ is bounded in $\{\alpha, \infty)$, its behaviour for $\sigma \to \infty$ is extremely simple in consequence of well-known theorems on almost periodic analytic functions. But also for unbounded functions, almost periodic in $\{\alpha, \infty)$, the situation is very perspicuous in view of the fact that the translation module F here always proves to be discrete, i. e. forms an arithmetic progression; in fact, a general "splitting theorem" holds which states that every such function $f(s)$ can be written (and essentially only in one way) as a sum of a function $p(s)$, unbounded and periodic in $\{\alpha, \infty)$, and a function $b(s)$, bounded and almost periodic in $\{\alpha, \infty)$. Moreover, also the converse is valid, viz. that the sum of a function $p(s)$, unbounded and periodic in $\{\alpha, \infty)$, and a function $b(s)$, bounded and almost periodic in $\{\alpha, \infty)$, is always almost periodic in $\{\alpha, \infty)$.

Chapter III deals with the case, where β is finite; here, we may suppose that $\beta = 0$ so that we have to do with functions which are almost periodic in a strip $\{\alpha, 0)$. The functions which are bounded and almost periodic in $\{\alpha, 0)$ are only shortly discussed, as the main object is the study of the functions unbounded in $\{\alpha, 0)$. For such an unbounded function, we begin proving that its translation module F cannot contain all real numbers. Subsequently, we distinguish between the case where the set F is discrete and that where F is everywhere dense. The first case, where F is an arithmetic progression, does not cause any difficulties; a general splitting theorem is valid here (completely analogous to that holding in the case of $\beta = \infty$), as $f(s)$ can be splitted—and practically uniquely—into a function $p(s)$ purely periodic in $\{\alpha, 0)$ and a function $b(s)$ bounded and almost periodic in $\{\alpha, 0)$. Next, the other (essentially more difficult) case is considered, where the translation module F is everywhere dense; in this case, the line $\sigma = 0$ is always an essentially singular line for the analytic function $f(s)$. The main question is, whether also here a general splitting theorem holds, analogous to that valid in the other cases. It is proved that this is not the case. In a decisive way we use a "gap theorem" concerning Dirichlet series $\sum a_n e^{\lambda_n s}$, convergent for $\sigma < 0$, where the exponents form an increasing sequence of positive numbers which increase "very strongly" to the infinite; this theorem states that the function $f(s)$ represented by such a series always is almost periodic in the whole strip $(-\infty, 0)$, and not only in $(-\infty, 0]$.

Finally, in Chapter IV, the proof of this gap theorem is given. Here, the treatment of Dirichlet series with strongly increasing exponents is extended somewhat further than necessary for the proper purpose of this paper. Thus, in order to throw light on the nature of the methods used, a new proof of a special case of the so-called HADAMARD gap theorem for Dirichlet series $\sum a_n e^{\lambda_n s}$ with the convergence half plane $\sigma < 0$

is given which states that the convergence line $\sigma = 0$ always is an essentially singular line for the analytic function represented by the series, if the exponents increase rapidly enough. In the case where the λ are integers (and we are concerned with a power series) this simple proof has already been communicated by the author in a paper[1] written in Danish.

[1] Om den Hadamard'ske "Hulsætning", Matematisk Tidsskrift, B (1919). See also the paper: Om Potensrækker med Huller, Matematisk Tidsskrift, B (1942).

CHAPTER II.

Functions almost periodic in a strip $\{\alpha, \infty)$.

We shall now investigate, how an analytic function $f(s)$ which is almost periodic in $\{\alpha, \infty)$ behaves for $\sigma \to \infty$.

We begin by recalling the following theorem which was proved in the third paper in Acta Mathematica cited above. If $f(s)$ is almost periodic in $\{\alpha, \infty]$, a necessary and sufficient condition that $f(s)$ is bounded for $\sigma \to \infty$, i.e. bounded in $\{\alpha, \infty)$, is that all the Dirichlet exponents Λ_n of the function are ≤ 0; if this condition is satisfied, the function is almost periodic not only in $\{\alpha, \infty]$, but also in $\{\alpha, \infty)$, and for $\sigma \to \infty$ uniformly in t it approaches a definite limit (viz. the constant term in the Dirichlet development of the function); in this case the function is called regular in the point $\sigma = \infty$. In view of a later application, we observe that the difference between two functions, bounded and almost periodic in $\{\alpha, \infty)$, again is almost periodic in $\{\alpha, \infty)$, and not only in $\{\alpha, \infty]$.

Thus, the functions almost periodic in a strip $\{\alpha, \infty)$, which are bounded for $\sigma \to \infty$, are simply the almost periodic functions regular in $\sigma = \infty$; an investigation is therefore only demanded in the case, where the function $f(s)$, almost periodic in $\{\alpha, \infty)$, is unbounded for $\sigma \to \infty$ (and, therefore, among its Dirichlet exponents has positive ones).

As the most simple type of such a function, we have obviously a function unbounded and purely periodic in $\{\alpha, \infty)$.

Let us begin by proving the following theorem: If $p(s)$ and $b(s)$ are two functions, almost periodic in $\{\alpha, \infty)$, of which $p(s)$ is purely periodic, while $b(s)$ is bounded in $\{\alpha, \infty)$, their sum

$$f(s) = p(s) + b(s)$$

is again almost periodic in $\{\alpha, \infty)$.

The theorem is not quite trivial, as it is not valid that the sum of any two functions almost periodic in (α, ∞) is again almost periodic in $\{\alpha, \infty)$ (but only in $\{\alpha, \infty])$; however, it is easy enough to prove. It is the task to show that to any arbitrarily given $\varepsilon > 0$ a relatively dense set of real numbers τ exists for which

$$|f(s+i\tau) - f(s)| \leqq \varepsilon \quad \text{in the whole strip} \quad \alpha \leqq \sigma < \infty.$$

As $b(s)$ tends to a limit for $\sigma \to \infty$ (uniformly in t), to the given ε we can first determine a number $\gamma = \gamma(\varepsilon)$ so that any real number τ is a translation number $\tau(\varepsilon)$ for the function $b(s)$ in the half plane $\{\gamma, \infty)$; from this, it follows that the function $b(s)$ has quite the same translation numbers $\tau(\varepsilon)$, whether we consider it in the half plane $\{\alpha, \infty)$ or in the strip $\{\alpha, \gamma\}$. In consequence of the earlier cited majorising property of the almost periodic functions $b(\sigma + it)$ $(\alpha \leqq \sigma \leqq \gamma)$ the set of translation numbers $\tau(\varepsilon)$ of $b(s)$ in $\{\alpha, \gamma\}$ has, however, a relatively dense intersection with any arithmetic progression—because, as is well-known, this is the case for the translation numbers of a single almost periodic function of a real variable. If, as difference in this arithmetic progression just $p > 0$ is chosen, where ip is a period of the given periodic function $p(s)$, any number τ in the previously mentioned relatively dense intersection is a translation number of the sum $f(s) = p(s) + b(s)$ in $\{\alpha, \infty)$, as

$$p(s+i\tau) = p(s) \quad \text{and} \quad |b(s+i\tau) - b(s)| \leqq \varepsilon \quad \text{in} \quad \alpha \leqq \sigma < \infty.$$

As we shall see, we have thereby actually exhausted all possibilities for a function almost periodic and unbounded in $\{\alpha, \infty)$, since the following inverse theorem is valid.

Splitting theorem: *Every function $f(s)$ unbounded and almost periodic in $\{\alpha, \infty)$ can—and essentially only in one way—be written as a sum*

$$f(s) = p(s) + b(s),$$

where $p(s)$ is periodic and unbounded in $\{\alpha, \infty)$, while $b(s)$ is bounded and almost periodic in $\{\alpha, \infty)$.

If we have a splitting of $f(s)$ as that stated in the theorem, and if ip is a period of the periodic term $p(s)$, the number p must necessarily belong to the translation module F of the function $f(s)$ in $\{\alpha, \infty)$, as in $\alpha \leqq \sigma < \infty$ the inequality

$$|f(s+ip) - f(s)| \leqq 2B$$

holds, where B means u.b. $|b(s)|$ in $\alpha \leqq \sigma < \infty$. It is, therefore, natural—and may also be interesting in itself—to study primarily this translation module F, i. e. to try to determine the numbers τ for which the difference $f(s+i\tau)-f(s)$ is bounded in $\{\alpha, \infty)$. Since the Dirichlet development of this difference is determined as the difference between the Dirichlet developments of $f(s+i\tau)$ and $f(s)$, i. e. given by

$$f(s+i\tau) - f(s) \sim \sum A_n (e^{i\Lambda_n \tau} - 1)\, e^{\Lambda_n s},$$

the theorem quoted in the beginning of this Chapter, however, involves immediately the following necessary and sufficient condition for the (anyhow in $\{\alpha, \infty]$) almost periodic function $f(s+i\tau)-f(s)$ to be bounded in $\{\alpha, \infty)$: In the Dirichlet series mentioned above must not occur any term with a positive exponent, i. e. for any positive Dirichlet exponent Λ_n of the given function $f(s)$ (necessarily occurring, because $f(s)$ is unbounded in $\{\alpha, \infty)$) it must be valid that $e^{i\Lambda_n\tau} - 1 = 0$, i. e.

$$\Lambda_n \tau \equiv 0 \pmod{2\pi} \quad \text{for any} \quad \Lambda_n > 0.$$

Thus, the numbers τ in the translation module F of $f(s)$ are just the numbers τ which are multipla of all the numbers $\frac{2\pi}{\Lambda_n}$, where Λ_n runs through the positive Dirichlet exponents of $f(s)$. Hereby (only applying that any τ has to be an integral multiple of one of these numbers $\frac{2\pi}{\Lambda_{n_0}}$) the translation module appears to be discrete, i. e. consists of all the numbers of an arithmetic progression νq $(q > 0,\ \nu = 0, \pm 1, \cdots)$. After this, turning to the exponents Λ_n, we notice (as $\tau = q$ satisfies all the congruences above) that all positive exponents Λ_n must be multipla of the number $\frac{2\pi}{q}$ and, further (as $\tau = q$ is the smallest positive solution of the congruences) that $\frac{2\pi}{q}$ is the greatest common divisor of the positive exponents Λ_n.

It is now easy to accomplish the proof of the splitting theorem, since it can be demonstrated that, as a period of the periodic term in a splitting of the desired kind, we may even use the number of least absolute value which may be taken into consideration, viz. the number iq, where q is the smallest positive number in the translation module F. As $p(s)$ we may use the periodic component (which was introduced in Chapter I) of $f(s)$ in $\{\alpha, \infty)$ belonging to the period iq. For, as the Dirichlet development of this function $p(s)$, purely periodic in $\{\alpha, \infty)$, consists of the terms $A_n e^{i\Lambda_n s}$ in the Dirichlet development of $f(s)$, for which Λ_n is a multiple of $\frac{2\pi}{q}$, the Dirichlet development of $p(s)$ coïncides as regards the terms with positive exponents (even with exponents ≥ 0) with the Dirichlet development of $f(s)$. Therefore, the Dirichlet development of the difference $b(s) = f(s) - p(s)$, almost periodic in $\{\alpha, \infty]$, can only contain terms with negative exponents; it follows that the function $b(s)$ actually is a function, bounded and almost periodic in $\{\alpha, \infty)$.

From the proof of the splitting theorem given above it is furthermore easy to decide to what degree the splitting is unique. Let us assume that

$$f(s) = p(s) + b(s)$$

is the "standard splitting" stated in the above proof, the period of $p(s)$ being the number iq where q is the smallest positive number in the translation module F, and $p(s)$ just being the periodic component of $f(s)$ belonging to that period iq; moreover, let us assume that

$$f(s) = p^*(s) + b^*(s)$$

is another arbitrary splitting of $f(s)$ in $\{\alpha, \infty)$ of the kind stated in the theorem. As for the period ip of the periodic term $p^*(s)$ it holds that p is a number of the translation module F, i. e. a number of the form νq, the difference $\pi(s) = p^*(s) - p(s) = b(s) - b^*(s)$ must be a function, bounded and periodic in $\{\alpha, \infty)$, with a period of the form $i\nu q$. Conversely, however, it is valid that, if $\pi(s)$ is a function bounded and periodic in $\{\alpha, \infty)$ with a period of the form $i\nu q$, we can use the function

$$p^*(s) = p(s) + \pi(s)$$

as a periodic splitting term; for, if we write $f(s)$ in the form

$$f(s) = (p(s) + \pi(s)) + (b(s) - \pi(s)),$$

the first term $p(s) + \pi(s)$ is periodic in $\{\alpha, \infty)$, while the other term $b(s) - \pi(s)$ is almost periodic and bounded in $\{\alpha, \infty)$, as the difference of two functions, almost periodic and bounded in $\{\alpha, \infty)$.

Thus it is clear that the splitting is "essentially" unique.

CHAPTER III.

Functions almost periodic in a strip $\{\alpha, \beta)$, where $\beta < \infty$.

Obviously it may be assumed, otherwise only applying the translation $s = s' + \beta$, that $\beta = 0$. In this chapter, the behaviour of a function, almost periodic in $\{\alpha, 0)$ $(-\infty < \alpha < 0)$, in the immediate neighbourhood to the left of the imaginary axis $\sigma = 0$ will be investigated.

Here, the situation is more complicated than in the case $\beta = \infty$, due to the fact that a function bounded and almost periodic in $\{\alpha, 0)$ does not show a similar simple behaviour for $\sigma \to 0$ as a function bounded and almost periodic in $\{\alpha, \infty)$ for $\sigma \to \infty$.

The functions, bounded and almost periodic in $\{\alpha, 0)$, however, shall be treated briefly, since we are especially interested in the unbounded functions. If $f(s)$ is bounded and almost periodic in $\{\alpha, 0)$ it has—already because it is bounded—according to a theorem by Fatou, for $\sigma \to 0$ a limit function $F(t) = f(it)$ in the sense that $f(\sigma + it)$ for $\sigma \to 0$ approaches a limit $F(t)$ for any t in $-\infty < t < \infty$ except in a set E of measure zero[1]. Thus, if $\tau = \tau(\varepsilon)$ is an arbitrary translation number of $f(s)$ in $\{\alpha, 0)$, i. e.

$$|f(s + it) - f(s)| \leqq \varepsilon \quad \text{in} \quad \alpha \leqq \sigma < 0,$$

[1] In its usual formulation Fatou's theorem deals with a function $f(z)$ bounded and analytic in a circle $|z| < 1$, and it states that, for almost all points z_0 on the boundary $|z| = 1$, $f(z)$ approaches a limit $f(z_0)$, if z tends to z_0 along the radius vector. When transforming the circle $|z| < 1$ by a linear substitution into the half plane $\sigma < 0$, we do not just arrive at the assertion mentioned in the text, as the radii vectori of the circle are not transformed into horizontal straight lines in the half plane, but into certain arcs of circles orthogonal to the boundary $\sigma = 0$. It is, however, obvious that a function $f(s)$, bounded and analytic in $\sigma < 0$, which approaches a limit g, if s approa-

we have for all t, except in a set of measure zero $E' = E'(\tau)$ (namely the sum of the set E and the set obtained from E by the translation $-\tau$), the inequality

$$|F(t+\tau) - F(t)| \leqq \varepsilon.$$

From this it follows especially that the limit function $F(t) = f(it)$ is a function almost periodic in STEPANOFF's sense, as for any of the mentioned translation numbers $\tau(\varepsilon)$ of $f(s)$ in $\{\alpha, 0)$ holds the inequality

$$\underset{-\infty < t < \infty}{\text{u. b.}} \int_t^{t+1} |F(t+\tau) - F(t)| \, dt \leqq \varepsilon.$$

However, we shall not study this limit function in detail, as such a study—analogous for instance to the study of a limit function of a function bounded and analytic in the unity circle $|z| < 1$—lies beyond the scope of this paper.

In order to obtain information on the different possibilities and to have some conveniently simple examples at our disposal —before turning towards our proper problem, viz. the investigation of the functions almost periodic and unbounded in $\{\alpha, 0)$— we shall, however, mention some typical examples of functions bounded and almost periodic in $\{\alpha, 0)$.

Example 1. It may, of course, happen that a function, bounded and almost periodic in $\{\alpha, 0)$, is almost periodic also in $\{\alpha, 0\}$, although it cannot be continued analytically across the line $\sigma = 0$. This is the case, for instance, with the purely periodic function $\sum_1^\infty \frac{1}{n^2} e^{n!s}$ $\left(\text{as the unity circle is the natural boundary of the power series } \sum \frac{1}{n^2} z^{n!}\right)$.

Example 2. The function

$$\varphi(s) = e^{\frac{e^s+1}{e^s-1}},$$

ches a point $s_0 = it_0$ of the boundary along such an arc of a circle, also converges to g, if s tends to s_0 along the tangent $t = t_0$, simply because the inequality $|f(s)| < K$ for $\sigma < 0$ involves that $f'(s) = O\left(\frac{1}{\sigma}\right)$ for $\sigma \to 0$, while the vertical segment between the tangent and the circle is $O(\sigma^2)$ for $\sigma \to 0$.

also bounded in the half plane $\sigma < 0$ and periodic with the period $2\pi i$, is regular on the whole boundary $\sigma = 0$ except the points $2\pi i m$ ($m = 0, \pm 1, \cdots$); that the function $\varphi(s)$ is bounded for $\sigma < 0$, viz. $|\varphi(s)| < 1$, follows from the fact that $|e^s| < 1$ for $\sigma < 0$ and the exponent $\frac{e^s + 1}{e^s - 1}$ therefore has a negative real part in $\sigma < 0$ (as by the linear function $u = \frac{w+1}{w-1}$ the unity circle $|w| < 1$ is transferred to the half plane $\Re(u) < 0$). If s ranges over the segment $\sigma = 0$ ($0 < t < 2\pi$), the function e^s ranges over the unity circle and therefore $\frac{e^s + 1}{e^s - 1}$ over the whole imaginary axis; thus $\varphi(s)$ will circulate an infinite number of times on the unity circle for $t \to 0$ and $t \to 2\pi$. We have called attention to this bounded function $\varphi(s)$, because, for any sufficiently small ε, in any case for $\varepsilon < 1$, it has no other translation numbers $\tau = \tau(\varepsilon)$ in $\{\alpha, 0)$ (α an arbitrary negative number) than just the numbers (periods) $2\pi m$. For, if τ is an arbitrary number $\neq 2\pi m$, in the difference

$$\varphi(s + i\tau) - \varphi(s)$$

the first term $\varphi(s + i\tau)$ converges to $\varphi(i\tau)$ for $s \to 0$, while the other term $\varphi(s)$ can be made to converge to an arbitrarily assigned value of the unity circle by letting s approach 0 in a convenient way from inside the half plane $\sigma < 0$; this implies, however, that $|\varphi(s + i\tau) - \varphi(s)|$ obtains values greater than ε in the half plane $\sigma < 0$ (by the way in every half circle $|s| < \delta$, $\sigma < 0$).

Example 3. Already in the introductory Chapter we mentioned two functions, almost periodic in $\{\alpha, 0)$, viz.

$$f_1(s) = \frac{1}{1 - e^s} \quad \text{and} \quad f_2(s) = \frac{1}{1 - e^{\sqrt{2}s}},$$

the sum of which is not almost periodic in $\{\alpha, 0)$; however, these functions are not bounded. It is of interest that the mentioned conditions can also occur for two functions $f_1(s)$ and $f_2(s)$ bounded and almost periodic in $\{\alpha, 0)$; hereby we have especially shown that there exist functions, for instance the sum $f_1(s) + f_2(s)$, which are almost periodic in $\{\alpha, 0]$ and bounded in $\{\alpha, 0)$, but not almost periodic in $\{\alpha, 0)$, in contrast to the case $\beta = \infty$. As an example we may use the two functions

$$f_1(s) = \varphi(s) \quad \text{and} \quad f_2(s) = \varphi(\sqrt{2}\,s),$$

where $\varphi(s)$ is the function of example 2. We realize that the sum $f(s) = f_1(s) + f_2(s)$, considered for instance in $\{-1, 0)$, has no translation number $\tau \neq 0$ for any $\varepsilon < 1$. In fact, if $\tau \neq 0$ is an arbitrarily given number, we can obviously choose a number t_0 in such a way that one and only one of the four numbers t_0, $t_0 + \tau$, $\sqrt{2}\,t_0$, $\sqrt{2}(t_0 + \tau)$ is a multiple of 2π (if τ itself is a multiple of 2π, we may, for instance, use $t_0 = \pi\sqrt{2}$; otherwise, we may use either the number $t_0 = 2\pi$ or, if $\sqrt{2}(2\pi + \tau)$ just is a multiple of 2π, the number $t_0 = 4\pi$). Therefore, if we make s approach the boundary point it_0 from the half plane $\sigma < 0$, three of the four functions $f_1(s)$, $f_1(s + i\tau)$, $f_2(s)$, $f_2(s + i\tau)$ will approach definite limits, while by making s tend to it_0 in a suitable way from the half plane $\sigma < 0$ we may arrive at any number on the unity circle as a limit of the fourth function; hence it is excluded (just as in example 2) that the modulus of the difference

$$f(s + i\tau) - f(s) = f_1(s + i\tau) + f_2(s + i\tau) - f_1(s) - f_2(s)$$

remains smaller than a number $\varepsilon < 1$ in the whole half plane $\sigma < 0$.

Now we begin the investigation of the functions, almost periodic and unbounded in $\{\alpha, 0)$. In order to demonstrate at once that the situation here is essentially different from that of the functions almost periodic and unbounded in $\{\alpha, \infty)$, it may be emphasized that the sum of a function $b(s)$, bounded and almost periodic in $\{\alpha, 0)$, and a function $p(s)$, unbounded and periodic in $\{\alpha, 0)$, is not necessarily almost periodic in $\{\alpha, 0)$. To this purpose, we need only consider the sum

$$f(s) = p(s) + b(s),$$

where $b(s)$ is the function $\varphi(s)$, bounded and almost periodic —even purely periodic—in $\{\alpha, 0)$, given in example 2, which has no other translation number for any $\varepsilon < 1$ than the numbers

$2\pi m$, while $p(s)$ is the periodic function $\frac{1}{1-e^{\sqrt{2}s}}$ with the period $\sqrt{2}\pi i$ and the poles $\sqrt{2}m\pi i$. For, if τ is a translation number of $f(s)$ in $\{\alpha, 0)$ (belonging to some ε or other), we must have $\tau = \sqrt{2}m\pi$ and, therefore, $p(s+i\tau) = p(s)$; but none of these numbers τ, except the trivial $\tau = 0$, is a translation number for $b(s)$ belonging to any $\varepsilon < 1$.

It is one of our main problems to decide to what extent a converse theorem would hold, i. e. to what degree we also here have a general splitting theorem analogous to that valid for $\beta = \infty$. As we shall see at the end of the Chapter, this is not always the case, although the theorem holds in an important special case.

Let us begin with the remark that a function $f(s)$, unbounded and almost periodic in $\{\alpha, 0)$, actually assumes values with arbitrarily large modulus already in a bounded part of the plane, precisely speaking, that there exists a length L such that the function is not bounded in any rectangle of the form $\alpha \leq \sigma < 0$, $t' < t < t' + L$. This is obviously valid for an L chosen thus that, in every interval $t' < t < t' + L$, there exists at least one translation number τ of the function $f(s)$ in $\{\alpha, 0)$ belonging for instance to $\varepsilon = 1$; for, if $f(s)$ was bounded (say $|f(s)| < K$) in only one of these rectangles, it would be bounded (viz. $|f(s)| < K+1$) in the whole strip $\alpha \leq \sigma < 0$. Hence, denoting a boundary point $s_0 = it_0$ as an "infinity point" of $f(s)$, if $f(s)$ is unbounded in every half circle $\sigma < 0$, $|s - s_0| < \delta$, we conclude, that $f(s)$ certainly has infinity points on the boundary $\sigma = 0$ (as $f(s)$ is bounded in every substrip $\alpha \leq \sigma \leq \gamma < 0$), and that these infinity points form a relatively dense set. That the existence of one infinity point s_0 involves the existence of a relatively dense set of infinity points, follows also from the fact that, together with s_0, in any case all points $s_0 + i\tau$ must be infinity points, where τ runs through the (relatively dense) translation module F of the function. The set of these infinity points of $f(s)$ on the boundary $\sigma = 0$ forms of course a closed set. This set can very well consist of all the points of the boundary; this is the case for the function $f(s) = \sum e^{n!s}$, purely periodic in $(-\infty, 0)$, where it holds for any fixed $t = 2\pi r$ (r rational) that $|f(\sigma + it)| \to \infty$ for $\sigma \to 0$.

For the more detailed study of a function, unbounded and almost periodic in $\{\alpha, 0)$, an investigation of the translation module F is of special importance. In the case $\beta = \infty$ considered in the foregoing chapter, the translation module was always discrete, i. e. it consisted of the numbers of an arithmetic progression. This may of course also happen when β is finite, for instance for the function $f(s) = \sum_{0}^{\infty} e^{ns} = \frac{1}{1-e^{s}}$ (purely periodic in $(-\infty, 0))$ with the poles $2m\pi i$, where F just consists of the numbers $2m\pi$. However, it may here also occur, that F is everywhere dense on the line $-\infty < t < \infty$. This is, for instance, the case for the above mentioned function $f(s) = \sum_{1}^{\infty} e^{n!s}$ (also purely periodic in $(-\infty, 0))$, since any rational multiple of 2π surely belongs to F, as for $\tau = \frac{m}{q} 2\pi$ in the whole half plane $\sigma < 0$ the inequality

$$|f(s+i\tau) - f(s)| = \sum_{1}^{q} \left| e^{n!(s+i\tau)} - e^{n!s} \right| \leq 2q$$

holds. In the case where the translation module F is everywhere dense, the infinity points must also be everywhere dense, as follows from a previous remark; therefore they must constitute the whole boundary (as the set is closed), and especially the boundary $\sigma = 0$ must always be an essentially singular line for the function.

The investigation of the functions, unbounded and almost periodic in $\{\alpha, 0)$, naturally falls into two cases corresponding to a discrete translation module and to an everywhere dense translation module, respectively. Before starting this investigation, it will first be proved that the translation module cannot consist of all real numbers. We observe that this is not a special case of a general theorem concerning arbitrary unbounded analytic functions, but a typical theorem for unbounded almost periodic functions; thus, the trivial not almost periodic analytic function $f(s) = s$ is unbounded in an arbitrary strip (α, β), while the difference $f(s+i\tau) - f(s)$ is bounded (in fact constant $= i\tau$) for every fixed τ.

Theorem. *The translation module of a function $f(s)$, unbounded and almost periodic in $\{\alpha, 0)$, cannot consist of all real numbers.*

It has to be proved that a function $f(s)$, almost periodic in $\{\alpha, 0)$, which has every number τ as translation number (for some or other ε), must necessarily be bounded.

A decisive step in the proof is the demonstration that the translation function

$$v(\tau) = \underset{s \text{ in } \{\alpha, 0)}{\text{u. b.}} |f(s+i\tau) - f(s)|,$$

here defined for all τ, is a bounded function of τ. To show this, it is obviously sufficient to prove that $v(\tau)$ is bounded in the interval $0 < \tau < L$, where L is chosen in such a way that any interval of the length L contains a number τ', which is a translation number of $f(s)$ in $\{\alpha, 0)$ corresponding to $\varepsilon = 1$; for if $v(\tau) < k$ in $0 < \tau < L$, due to the inequality $v(\tau_1 + \tau_2) \leqq v(\tau_1) + v(\tau_2)$, the function $v(\tau)$ is obviously $< k+1$ for all τ. That the translation function $v(\tau)$ is bounded in a finite interval $0 < \tau < L$ can be demonstrated in the following way: We consider $v(\tau)$ in the interval $-L < \tau < L$ and denote by E_n the (measurable) set in $-L < \tau < L$ in whose points $v(\tau) < n$. As $E_1 \subseteqq E_2 \subseteqq E_3 \subseteqq \cdots$ and as any τ in $-L < \tau < L$ belongs to the set E_n for n sufficiently large, the measure $m(E_n)$ of the set E_n will converge to the length $2L$ of the whole interval, for $n \to \infty$. Therefore, we may determine N such that

$$m(E_N) > \frac{3}{2} L.$$

Then, for any τ in $0 < \tau < L$ the inequality

$$v(\tau) < 2N$$

will be valid. This is evidently proved, if we have shown that any given τ in $0 < \tau < L$ can be written in the form $\tau = \tau_1 - \tau_2$, where both τ_1 and τ_2 belong to the set E_N (and τ_1 lies in the interval $0 < \tau < L$). This is possible, as a consequence of the fact that the set E_N and the set $E'_N = E'_N(\tau)$ obtained from E_N by translating it by τ must have a common point between 0 and L (which point then is equal to τ_1 as well as to $\tau + \tau_2$, where τ_1 and τ_2 both belong to E_N), the set E_N as well as the set E'_N having an intersection with the interval $0 < \tau < L$ the measure of which is greater than $\frac{L}{2}$.

Now it is easy to finish the proof, viz. to show that $f(s)$ must be bounded in $\{\alpha, 0)$. We have only to choose an arbitrary number Λ which is incommensurable with all the Dirichlet exponents of $f(s)$ different from zero (forming a countably infinite set) and to consider the periodic component $p(s)$ of $f(s)$ in $\{\alpha, 0)$ belonging to the period $ip = \frac{2\pi}{\Lambda}i$. As no Dirichlet exponent of $f(s)$ which is different from zero is a multiple of Λ, this periodic component is simply a constant c, viz. the constant term in the Dirichlet development of $f(s)$. On the other hand, however, $p(s)$ can be determined (in the whole strip $\alpha \leqq \sigma < 0$) by

$$p(s) = \lim_{n \to \infty} \frac{f(s+ip) + f(s+2\,ip) + \cdots + f(s+nip)}{n};$$

hence, it results that the difference $p(s) - f(s)$ must be bounded in the whole strip $\{\alpha, 0)$, viz. that its modulus must be $\leqq G =$ u. b. $v(\tau)$, as we have

$$p(s) - f(s) =$$

$$\lim_{n \to \infty} \frac{(f(s+ip) - f(s)) + (f(s+2\,ip) - f(s)) + \cdots + (f(s+nip) - f(s))}{n},$$

where the modulus of each of the differences $f(s+i\nu p) - f(s)$ is $\leqq G$ in $\{\alpha, 0)$. From $p(s) = c$ and $|p(s) - f(s)| \leqq G$ in $\{\alpha, 0)$, it finally results that

$$|f(s)| \leqq |c| + G \quad \text{in} \quad \{\alpha, 0)$$

and so we have proved that $f(s)$ is bounded.

In the following we shall first treat the case where the translation module F is discrete, and then the case where F is everywhere dense on the line $-\infty < t < \infty$.

The translation module is discrete.

In this case the situation proves to be highly analogous to that for $\beta = \infty$ (where the translation module always is discrete), as the following splitting theorem is valid.

Splitting theorem: *A function $f(s)$, unbounded and almost periodic in $\{\alpha, 0)$, with a discrete translation module can always—and practically in one way only—be written as a sum*

$$f(s) = p(s) + b(s),$$

where $p(s)$ is a function, unbounded and purely periodic in $\{\alpha, 0)$, while $b(s)$ is a function, bounded and almost periodic in $\{\alpha, 0)$.

Let us denote the numbers in the translation module F which, in consequence of the assumption, form an arithmetic progression, by $\tau = \nu q \, (q > 0, \ \nu = 0, \pm 1, \cdots)$. It is obvious that, in any splitting of $f(s)$ of the kind mentioned in the theorem, each period of the periodic term must necessarily have the form $i\nu q$. It will be proved that, as a period ip of the periodic term, we may even use the number iq, where q is the smallest positive number which might be taken into consideration, for we shall prove (in analogy to the case $\beta = \infty$) that the periodic component of $f(s)$ in $\{\alpha, 0)$ belonging to the period iq is a possible $p(s)$. When proving this, it would not be convenient (as in the case $\beta = \infty$) to use the Dirichlet developments, because here (in contrast to the case $\beta = \infty$) we have no simple criterion, whether a Dirichlet development just represents a function bounded in $\{\alpha, 0)$. We have to operate with $p(s)$, determined in $\{\alpha, 0)$ as a mean value, i. e. by the limit equation

$$p(s) = \lim_{n \to \infty} \frac{f(s + iq) + f(s + 2\,iq) + \cdots + f(s + niq)}{n}.$$

We have to show that the function $b(s)$—obtained by subtracting from $f(s)$ this function $p(s)$ of the period iq, purely periodic in $\{\alpha, 0)$—is not only (of course) almost periodic in $\{\alpha, 0]$, but actually almost periodic in $\{\alpha, 0)$ and moreover is bounded in $\{\alpha, 0)$.

However, it is plain that the difference $b(s) = f(s) - p(s)$ is almost periodic in the whole strip $\{\alpha, 0)$. For, as any translation number τ of $f(s)$ in $\{\alpha, 0)$ lies in the translation module F, i. e. has the form νq, the number $i\tau$ is a period of $p(s)$, and therefore any translation number $\tau(\varepsilon)$ of $f(s)$ in $\{\alpha, 0)$ is also a translation number $\tau(\varepsilon)$ of $b(s)$ in $\{\alpha, 0)$.

In order to prove that the function $b(s) = f(s) - p(s)$ is bounded in $\{\alpha, 0)$ we show primarily that the translation function $v(\tau)$ of $f(s)$ in $\{\alpha, 0)$, which here is defined for $\tau = \nu q$ $(\nu = 0, \pm 1, \cdots)$ only, is bounded, i. e. that

$$v(\tau) \leqq K \quad \text{for all} \quad \tau = \nu q \ (\nu = 0, \pm 1, \cdots).$$

To this purpose we consider the set of translation numbers τ of $f(s)$ belonging, for instance, to $\varepsilon = 1$. As each of these numbers is a multiple of q, the relative density of the set formed by these numbers $\tau(1)$ implies that there exists a positive integer M such that among M arbitrary consecutive multiples of q there exists at least one which is a $\tau(1)$. If now k denotes the greatest of the M numbers $v(\nu q)$ $(\nu = 1, 2, \cdots M)$, for all $\tau = \nu q$ the inequality $v(\tau) \leqq k + 1$ obviously holds. Having thus proved the inequality $v(\tau) \leqq K$ for all $\tau = \nu q$, it is plain that the function $b(s)$ is bounded in the whole strip $\{\alpha, 0)$, since for any point s in $\{\alpha, 0)$ the limit equation

$$b(s) = f(s) - p(s) =$$

$$\lim_{n \to \infty} \frac{(f(s) - f(s + iq)) + (f(s) - f(s + 2\,iq)) + \cdots + (f(s) - f(s + inq))}{n}$$

is valid, where the modulus of each occurring difference

$$f(s) - f(s + i\nu q)$$

is $\leqq K$; hence also $|b(s)| \leqq K$.

Subsequently, it is easy to decide to what degree the mentioned splitting is unique and, as we shall see, the result is quite analogous to that found for $\beta = \infty$. Let

$$f(s) = p(s) + b(s)$$

be the "standard splitting" given in the proof above, in which $p(s)$ is the periodic component of $f(s)$ belonging to the period iq, where q is the smallest positive number in the translation module F, and let

$$f(s) = p^*(s) + b^*(s)$$

be an arbitrary splitting of $f(s)$ in $\{\alpha, 0)$ of the kind mentioned in the theorem. As the periodic term $p^*(s)$ certainly has a number of the form imq as a period (m a positive integer), the difference $\pi(s) = p^*(s) - p(s) = b(s) - b^*(s)$ must necessarily be a function, bounded and periodic in $\{\alpha, 0)$ with a period of the form imq. Conversely, however, it also holds that for any function $\pi(s)$, bounded and periodic in $\{\alpha, 0)$, with a period of the form imq, we may use as a periodic splitting term the function

$$p^*(s) = p(s) + \pi(s)$$

periodic in $\{\alpha, 0)$, i. e. the function

$$b^*(s) = b(s) - \pi(s)$$

is not only (of course) bounded in $\{\alpha, 0)$ and almost periodic in $\{\alpha, 0]$, but also almost periodic in the whole strip $\{\alpha, 0)$. Evidently, this is proved, when we have shown that for any $\varepsilon > 0$ the function $b(s)$ has in $\{\alpha, 0)$ a relatively dense set of translation numbers (not only, as we already know, of the form νq, but also) of the form $\nu m q$. To see this, we only need to apply that $b(s)$ has a relatively dense set of translation numbers of the form νq belonging to $\frac{\varepsilon}{m}$, and that these latter translation numbers multiplied by m are translation numbers of $b(s)$ belonging to ε itself.

The translation module F is everywhere dense.

We have already mentioned that there exist functions $f(s)$, unbounded and almost periodic in $\{\alpha, 0)$, whose translation module is everywhere dense on the line $-\infty < t < \infty$; the periodic function $f(s) = \sum_1^\infty e^{n!s}$ considered in $\{-1, 0)$, for instance, is of this type. Furthermore, we have seen that any function $f(s)$ of this type has the boundary $\sigma = 0$ as an essentially singular line, and even that every point on the boundary is an infinity point. Our main result concerning these functions is comprised in the following (negative) theorem:

Theorem. *There exist functions $f(s)$, almost periodic and unbounded in a strip $\{\alpha, 0)$, which cannot be splitted into a sum*

$$f(s) = p(s) + b(s),$$

where $p(s)$ is purely periodic in $\{\alpha, 0)$, while $b(s)$ is bounded and almost periodic in $\{\alpha, 0)$.

Moreover, we shall prove the somewhat further going theorem that such a splitting is not always possible, even if we only demand that the function $b(s)$ (which on account of the equality $b(s) = f(s) - p(s)$ automatically is almost periodic in $\{\alpha, 0])$ be bounded in $\{\alpha, 0)$, but not that it be almost periodic in the whole strip $\{\alpha, 0)$.

In order to construct a "counter-example" which is suited to prove the correctness of the assertion made in the theorem, we shall first look for a general type of examples concerning functions unbounded and almost periodic in a strip $\{\alpha, 0)$ and having an everywhere dense translation module. Starting from the simple example $f(s) = \sum_1^\infty e^{n!s}$ (periodic and therefore quite unadapted to our proper purpose), it is obvious to think of almost periodic functions with Dirichlet exponents which form a sequence strongly increasing to the infinity, i. e. ordinary Dirichlet series

$$f(s) = \sum_1^\infty a_n e^{\lambda_n s} \quad (0 < \lambda_1 < \lambda_2 < \cdots \to \infty)$$

with so-called "gaps", i. e. with very large intervals between the exponents. Actually, the general theorem holds that every such series represents a function $f(s)$, unbounded and almost periodic in $(-\infty, 0)$ (and not only in $(-\infty, 0])$, with an everywhere dense translation module, if the series has the half plane $\sigma < 0$ as convergence half plane and is divergent on the boundary line $\sigma = 0$. We shall postpone the formulation of this "gap theorem" and its proof to the next Chapter. Here, we shall confine ourselves—as this is sufficient for our present purpose—to mentioning that it results from this gap theorem that every Dirichlet series $\sum e^{\lambda_n s}$ with coefficients $a_n = 1$ for which

$$0 < \lambda_1 < \lambda_2 < \cdots \to \infty \quad \text{and} \quad \lambda_{n+1} > e^{\lambda_n}$$

represents a function $f(s)$, unbounded and almost periodic in $(-\infty, 0)$, and, therefore, also for instance in $\{-1, 0)$, with an everywhere dense translation module.

Within this class of functions $f(s)$ we shall attempt to determine one which may be called highly "aperiodic" in the sense that by splitting off a function $p(s)$ purely periodic in $\{-1, 0)$ we can never obtain a function $b(s)$ bounded in $\{-1, 0)$.

As we shall see, we have a function of this type in every function $f(s)$ of our class with rationally independent exponents λ_n. The task is to show that such a function $f(s)$ cannot be written in the form

$$f(s) = p(s) + b(s),$$

where $p(s)$ is purely periodic in $\{-1, 0)$ and $b(s)$ bounded in $\{-1, 0)$ (and of course almost periodic in $\{-1, 0])$. We give an indirect proof and, consequently, suppose that such a representation exists. As the Dirichlet exponents of the function $f(s)$ are rationally independent, at the most one of them can have the form $\frac{2\pi}{p}\nu$, where ip is a period of $p(s)$. In the following we may assume that $f(s)$ has no Dirichlet exponents of this form, as, if $\Lambda = \frac{2\pi}{p}\nu$ was such a Dirichlet exponent, we should only subtract $e^{\Lambda s}$ on both sides of the equation, exactly speaking we should replace $f(s)$ by $f(s) - e^{\Lambda s}$, and $b(s)$ by $b(s) - e^{\Lambda s}$, whereby $f(s) - e^{\Lambda s}$ as $f(s)$ is unbounded in $\{-1, 0)$, and $b(s) - e^{\Lambda s}$ as $b(s)$ is bounded in $\{-1, 0)$. We now write the equation $f(s) = p(s) + b(s)$ in the form

$$b(s) = -p(s) + f(s)$$

and we consider this equation in the strip $\{-1, -\varepsilon)$ where $0 < \varepsilon < 1$. The periodic term $-p(s)$ is obviously just the periodic component of $b(s)$ in $\{-1, -\varepsilon)$ belonging to the period ip, because $f(s)$ has no Dirichlet exponents of the form $\frac{2\pi}{p}\nu$. Hence, we have the inequality

$$\underset{s \text{ in } \{-1, -\varepsilon)}{\text{u. b.}} |p(s)| \leq \underset{s \text{ in } \{-1, -\varepsilon)}{\text{u. b.}} |b(s)|.$$

Consequently, if B denotes upper bound of $|b(s)|$ in the whole strip $\{-1, 0)$, the absolute value of $p(s)$ is $\leq B$ in the whole strip $\{-1, 0)$. However, this contradicts the fact that $f(s)$ is unbounded in $\{-1, 0)$, as it would involve that in the whole strip $\{-1, 0)$ the inequality

$$|f(s)| \leq |p(s)| + |b(s)| \leq 2B$$

was valid. Hereby, the proof of our theorem is fulfilled.

CHAPTER IV.

A gap theorem concerning the almost periodicity of Dirichlet series.

In this Chapter, we shall only deal with Dirichlet series in the classical sense, i. e. with series

$$\sum_1^\infty a_n e^{\lambda_n s}, \text{ where } 0 < \lambda_1 < \lambda_2 < \cdots \to \infty .$$

We shall even consider such series only, whose exponents increase strongly to the infinite, from which it follows in particular that the series is absolutely convergent in the whole convergence half plane of the series, which may be supposed to be the half plane $\sigma < 0$. Then, for $\sigma < 0$, the series represents an analytic function $f(s)$ which is almost periodic in $(-\infty, 0]$ and has the given series as its Dirichlet development.

The so-called HADAMARD's gap theorem for Dirichlet series states, generally speaking, that the convergence line $\sigma = 0$ always is an essentially singular line of the function $f(s)$ represented by the series, if the sequence of the exponents increases rapidly enough. In order to illustrate the kind of reflections made below by an especially simple case, we shall begin this Chapter by proving HADAMARD's gap theorem in a rather extreme case, viz. the case where the exponents increase so strongly that the ratio of an exponent and the foregoing one is greater than a constant > 3. As we shall see, the theorem can then be proved in a particularly simple way.

A special case of Hadamard's gap theorem. *A function $f(s)$ represented by a Dirichlet series $\sum a_n e^{\lambda_n s}$ with the convergence half plane $\sigma < 0$ whose exponents satisfy the inequality*

$$\frac{\lambda_{n+1}}{\lambda_n} > k > 3 \quad (\text{for } n > n_0)$$

has the convergence line $\sigma = 0$ *as an essentially singular line. If* $\sum |a_n|$ *is divergent, all the points of the line* $\sigma = 0$ *are infinity points of* $f(s)$.

The proof is based on the following well-known (and easily provable) theorem of VIVANTI and LANDAU: If for a Dirichlet series $f(s) = \sum a_n e^{\lambda_n s}$ with the convergence half plane $\sigma = 0$ it is valid that in a point $s_0 = it_0$ of the boundary all the terms $a_n e^{i\lambda_n t_0}$ are positive from a certain step, the point s_0 is always a singular point of the function $f(s)$—whether the series is convergent or divergent in the point. We shall use this theorem in the well-known, somewhat more comprehensive formulation where the assumption that all the terms $a_n e^{i\lambda_n t_0}$ are positive from a certain step is replaced by the weaker assumption that all the terms $a_n e^{i\lambda_n t_0}$ from a certain step lie in a fixed angle $< \pi$, for instance in an angle $-\frac{\pi}{2} + d < v < \frac{\pi}{2} - d$ $\left(\text{where } 0 < d < \frac{\pi}{2}\right)$. Moreover, we shall use the following simple remark: if on the line $-\infty < t < \infty$ there lie intervals of a fixed length $\beta < \alpha$, periodically with a period $\alpha > 0$, then every interval with a length $> \alpha + \beta$ in its interior contains at least one of the mentioned intervals of the length β.

In order to prove that all points of the line $\sigma = 0$ are singular points of the function $f(s)$ it is, of course, sufficient to prove that the singular points lie everywhere dense on the line, that is to say that there exists a singular point in every interval $t_1 < t < t_2$ on the line $\sigma = 0$. In consequence of the theorem mentioned above, this is certainly the case, if in the arbitrarily given interval $t_1 < t < t_2$ there exists a point t_0 such that all the terms $a_n e^{i\lambda_n t_0}$ from a certain step lie in the fixed angle $-\frac{\pi}{2} + d < v < \frac{\pi}{2} - d$, where we assume d chosen so small (which is possible because of the inequality $\frac{\lambda_{n+1}}{\lambda_n} > k > 3$ for $n > n_0$) that

$$\frac{\pi - 2d}{\lambda_n} > \frac{3\pi - 2d}{\lambda_{n+1}} \quad \text{for} \quad n > n_0.$$

Concerning the n-th term $a_n e^{i\lambda_n t}$, the intervals I_n on the t-axis, in whose points the amplitude of the term lies in the angle $-\frac{\pi}{2}+d<v<\frac{\pi}{2}-d$, have the length $\frac{\pi-2d}{\lambda_n}$, and they repeat themselves periodically with the period $\frac{2\pi}{\lambda_n}$. In consequence of a remark above, every interval of a length greater than

$$\frac{2\pi}{\lambda_n}+\frac{\pi-2d}{\lambda_n}=\frac{3\pi-2d}{\lambda_n}$$

therefore certainly contains one of the mentioned intervals I_n. By virtue of the inequality written above, for $n>n_0$ each interval I_n contains an interval I_{n+1}. Now, we can immediately complete the proof. We have only to choose $N>n_0$ so great that the given interval $t_1<t<t_2$ contains an interval I_N. Inside this interval we have then to determine an interval I_{N+1}, inside that again an interval I_{N+2} etc. If t_0 denotes the common point of the sequence of intervals thus determined, all the terms $a_n e^{i\lambda_n t_0}$ for $n \geqq N$ are situated in the angle $-\frac{\pi}{2}+d<v<\frac{\pi}{2}-d$, and the point $s_0 = it_0$ is therefore a singular point of $f(s)$.

In the case where $\sum |a_n|$ is divergent, it is moreover clear that the point $s=it_0$ thus obtained is an infinity point of $f(s)$ (as the above consideration shows that $|f(\sigma+it_0)|\to\infty$ for $\sigma\to 0$); thus, if $\sum |a_n|$ is divergent, in every interval $t_1<t<t_2$ there exist infinity points of $f(s)$, i. e. the boundary consists of nothing but infinity points.

We shall now formulate and prove the main theorem of this chapter.

An almost periodic gap theorem. *If $\lambda_1<\lambda_2<\cdots$ is a sequence of positive numbers, which (for the sake of simplicity) we shall suppose to be >1, and which are increasing so strongly to the infinite that*

$$\lambda_{n+1}>e^{k\lambda_n} \quad \textit{for all} \quad n,$$

where k is a positive constant, then every Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n s}$$

belonging to this sequence of exponents and convergent for $\sigma < 0$ *—and therefore also absolutely convergent for* $\sigma < 0$*—represents a function* $f(s)$ *analytic in* $\sigma < 0$ *which is almost periodic in the whole half plane* $(-\infty, 0)$ *and not only in* $(-\infty, 0]$.

Moreover, the translation module F *of* $f(s)$ *in* $(-\infty, 0)$ *is everywhere dense on the* t*-axis, i. e. in every interval* $t_1 < t < t_2$ *exists a number* τ *such that the difference* $f(s+i\tau) - f(s)$ *is bounded in the whole half plane* $\sigma < 0$.

If $\sum |a_n|$ is convergent, the theorem is trivial, and therefore we may suppose in the proof that $\sum |a_n|$ is divergent; as a consequence of a remark given above it is then valid—as the condition which is now imposed on the exponents λ_n is much stronger than the former one—that the boundary $\sigma = 0$ consists of nothing but infinity points of the function $f(s)$.

Proof: We choose a fixed positive $c < k$. As the series $\sum a_n e^{-\lambda_n c}$ is convergent, there exists a constant K such that $|a_n| e^{-\lambda_n c} < K$ for all n; we may suppose $K = 1$ (otherwise we only divide all the coefficients a_n by K), i. e.

$$|a_n| < e^{\lambda_n c} \quad \text{for all } n.$$

Next, we choose a number c' such that

$$c < c' < k$$

and write

$$\varepsilon_n = e^{-\lambda_n c'} \quad \text{for all } n.$$

Especially it is valid that the series $\sum \varepsilon_n e^{\lambda_n c}$ and thereby a fortiori the series $\sum \varepsilon_n |a_n|$ is convergent.

Let us consider the exponential factor $e^{i\lambda_n t}$. Periodically with a period $\frac{2\pi}{\lambda_n}$, on the t-axis there lie intervals I_n of the length $\frac{2\varepsilon_n}{\lambda_n}$ in whose points t the exponent $\lambda_n t$ differs from an integral multiple of 2π less than ε_n; in these points it is furthermore valid that

$$|e^{i\lambda_n t} - 1| < \varepsilon_n.$$

For large n, say for $n > n_0$, every one of our intervals I_n contains one of our intervals I_{n+1}, as for large n the inequality

$$\frac{2\,\varepsilon_n}{\lambda_n} > \frac{2\,\pi}{\lambda_{n+1}} + \frac{2\,\varepsilon_{n+1}}{\lambda_{n+1}}$$

is valid; this is evident, as the right side is $< \frac{4\,\pi}{\lambda_{n+1}}$ and as for large n (because $c' < k$) we have

$$\lambda_{n+1} > \frac{2\,\pi}{\varepsilon_n}\,\lambda_n = 2\,\pi\lambda_n e^{\lambda_n c'}.$$

From this, one of the assertions made in the theorem follows at once, viz. that in every given interval $t_1 < t < t_2$ there exists a number τ for which the difference $f(s+i\tau)-f(s)$ is bounded in the whole half plane $\sigma < 0$. We have only to determine $N > n_0$ so large that the interval $t_1 < t < t_2$ contains one of our intervals I_N; then, we determine in it an interval I_{N+1}; in it again an interval I_{N+2} etc. The common point of the sequence of intervals $I_N, I_{N+1}, I_{N+2}, \cdots$ is then a point in $t_1 < t < t_2$ for which $f(s+i\tau)-f(s)$ is bounded in $\sigma < 0$, as for every s in $\sigma > 0$ the inequality holds

$$\begin{aligned} |f(s+i\tau)-f(s)| &\leq 2\sum_{n=1}^{N-1}|a_n| + \sum_{n=N}^{\infty}|a_n|\cdot|e^{i\lambda_n\tau}-1| \\ &\leq 2\sum_{n=1}^{N-1}|a_n| + \sum_{n=N}^{\infty}|a_n|\,\varepsilon_n < \infty. \end{aligned}$$

However, we must proceed somewhat more cautiously when proving our main assertion, viz. that $f(s)$ is almost periodic in $(-\infty, 0)$, i.e. that to any arbitrarily given ε, which we may suppose to be < 1, there exists a length $L = L(\varepsilon)$ such that, in every interval of the length L, there exists a number τ for which

$$|f(s+i\tau)-f(s)| \leq \varepsilon \quad \text{in the whole half plane} \quad \sigma < 0.$$

We determine a number $N = N(\varepsilon) > n_0$ such that

$$\sum_{N+1}^{\infty}|a_n|\,\varepsilon_n < \frac{\varepsilon}{2} \quad \text{and} \quad \lambda_{N+1} > \frac{16\,\pi}{\varepsilon}\,N\lambda_N e^{\lambda_N c}.$$

We can fulfill the last condition, because $c < k$ and $\lambda_{n+1} > e^{k\lambda_n}$. Now we split the function $f(s)$ in the half plane $\sigma < 0$ into a beginning $B_N(s)$ and a remainder $R_N(s)$, namely

$$B_N(s) = \sum_1^N a_n e^{\lambda_n s} \quad \text{and} \quad R_N(s) = \sum_{N+1}^{\infty} a_n e^{\lambda_n s}.$$

Here, the function $B_N(s)$ is of course almost periodic in $(-\infty, 0)$, namely even almost periodic in $(-\infty, \infty]$. We determine a length L (>1) such that every interval of the length L contains a translation number τ of $B_N(s)$ in $(-\infty, 0)$ belonging to $\frac{\varepsilon}{4}$. Then, this length will be a usable length $L(\varepsilon)$ of $f(s)$ in $(-\infty, 0)$. To prove this we primarily estimate the differential coefficient $B_N'(s)$ in $\sigma < 0$ and find

$$|B_N'(s)| = |\sum_{n=1}^N a_n \lambda_n e^{\lambda_n s}| \leq \sum_{n=1}^N \lambda_n |a_n| \leq N \lambda_N e^{\lambda_N c}.$$

If we set

$$l_N = \frac{\frac{\varepsilon}{4}}{N \lambda_N e^{\lambda_N c}},$$

it holds for every t_0 in the interval $\tau - l_N < t < \tau + l_N$ around one of our translation numbers $\tau = \tau\left(\frac{\varepsilon}{4}\right)$ of $B_N(s)$ in $(-\infty, 0)$, that in $(-\infty, 0)$

$$|B_N(s + it_0) - B_N(s + i\tau)| \leq l_N N \lambda_N e^{\lambda_N c} = \frac{\varepsilon}{4};$$

consequently the interval $\tau - l_N < t < \tau + l_N$ consists of nothing but translation numbers of $B_N(s)$ in $(-\infty, 0)$ belonging to $\frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$. In particular, we find $\left(\text{as } L > 1 \text{ and } l_N < \frac{1}{2}\right)$ that every interval $\gamma < t < \gamma + L$ of the length L contains a whole interval i_N of the length l_N whose points all are translation numbers of $B_N(s)$ in $(-\infty, 0)$ belonging to $\frac{\varepsilon}{2}$. However, we have

$$l_N = \frac{\varepsilon}{4 N \lambda_N e^{\lambda_N c}} > \frac{4\pi}{\lambda_{N+1}},$$

and our interval i_N of the length l_N contains therefore at least one of the intervals I_{N+1} mentioned above. Now we proceed as before; we determine an interval I_{N+2} inside I_{N+1}, in it again an interval I_{N+3}, and so on. The common point of the sequence of intervals i_N, I_{N+1}, I_{N+2}, $\cdots$ is called τ. This number τ lies in the interval $\gamma < t < \gamma + L$, and it is further a translation number of $f(s)$ in $(-\infty, 0)$ belonging to ε. In fact, the inequality $|e^{i\lambda_n t} - 1| < \varepsilon_n$ is valid for t lying in an interval I_n; hence for $\sigma < 0$ we get

$$|f(s+i\tau)-f(s)| \leqq |B_N(s+i\tau)-B_N(s)|+|R_N(s+i\tau)-R_N(s)|$$

$$\leqq \frac{\varepsilon}{2}+\sum_{N+1}^{\infty}|a_n|\,|e^{i\lambda_n\tau}-1| \leqq \frac{\varepsilon}{2}+\sum_{N+1}^{\infty}|a_n|\,\varepsilon_n$$

$$\leqq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Thus, the theorem is established.

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